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ELECTROMAGNETIC MOMENTUM DENSITY AND RADIATION PRESSURE IN NORMALLY DISPERSIVE, NON-DISSIPATIVE MEDIA

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Consideration of the electromagnetic momentum density **D x B** associated with a parallel plate capacitor in the presence of a magnetic field parallel to the plates shows that the momentum is a property of the electromagnetic field in a dielectric (closely related to the polarization of the dielectric) rather than a matter of storing mechanical momentum in the dielectric. This has implications regarding transport of momentum by electromagnetic waves. The physical cause of group velocity in normally dispersive, non-dissipative media is the immobilization of part of the wave energy, the energy transport being due to just part of the energy density propagating at the phase velocity. Something equivalent occurs in the case of momentum flow, and the existence of non-propagating momentum is another characteristic of group velocity. In idealized dielectrics, all of the electromagnetic momentum associated with a wave field flows at the phase velocity; the non-propagating momentum involves an impulse in the direction of propagation delivered to the dielectric upon the establishment of the wave field, and an impulse in the opposite direction delivered to the dielectric upon its decay. Although this appears to describe storage of momentum in the dielectric, the second momentum transfer occurs whether or not the momentum delivered to the dielectric upon (OVER)

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establishment of the wave field is retained by the dielectric for the duration of the wave field. In plasmas, part of the electromagnetic momentum does not propagate with the waves, and the picture is more complicated than it is for dielectrics.

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INTRODUCTION

For an electromagnetic wave field in a simple medium with dispersion, only part of the energy associated with the wave field propagates with the waves; the remainder does not, although it is converted to propagating energy when the wave field decays [Johnson, 1991]. By simple media, we mean linear, isotropic, homogeneous media exhibiting normal dispersion without absorption at the frequencies under consideration. As the flow of momentum is directly related to the flow of energy, something similar must be true of the momentum associated with the wave field, giving rise to the requirement that some of the momentum associated with the wave field not propagate with the waves. Still, the accounting for momentum in electromagnetic waves is a trickier business than the accounting for energy [Stedman, 1992]. We will investigate the propagation of electromagnetic momentum density (emmd) in wave fields in simple media, but it will be helpful to consider first the properties of momentum density in two static situations, namely a charged particle near a solenoid and a charged parallel-plate capacitor with a magnetic field oriented parallel to the plates. The former will illustrate the relationships between emmd and the vector potential of a current distribution [The terminology vector potential of a current distribution was suggested by Cullwick (1959) as preferable to vector potential of the magnetic field.] and the latter will illustrate the relationships between electromagnetic and mechanical momentum and between emmd and displacement currents in simple media.

VECTOR POTENTIAL AND ELECTROMAGNETIC MOMENTUM

Thomson [1904a] identified the angular momentum L associated with a configuration consisting of magnetic monopole m and an electric charge q as

$$\mathbf{L} = \hat{\mathbf{r}}\ \mu_0 \mathrm{qm}/4\pi, \tag{1}$$

where $\hat{\mathbf{r}}$ is the unit vector directed from q to m. Thomson evaluated the angular momentum as

$$\mathbf{L} = \int \mathbf{r} \times (\epsilon_0 \mathbf{E} \times \mathbf{B}) \mathrm{d}\tau, \tag{2}$$

where the integration is over all space. To reproduce Thomson's result, place q and m at –b/2 and b/2 on the z axis, as illustrated in Figure 1. $\mathbf{E} \times \mathbf{B}$ is azimuthal about z, so all volume elements contribute to angular momentum in the +z direction, and

$$\begin{aligned}
\mathrm{L} &= \int \rho \epsilon_0 \mathrm{EB} \sin\theta \, \mathrm{d}\tau = \epsilon_0 \int \rho \frac{\mathrm{q}}{4\pi\epsilon_0 \mathrm{r}_1{}^2} \frac{\mu_0 \mathrm{m}}{4\pi \mathrm{r}_2{}^2} \sin(\theta_2 - \theta_1) \, \mathrm{d}\tau \\
&= \frac{\mu_0 \mathrm{qm}}{4\pi} \int_0^\infty \int_0^\infty \rho^2 \frac{\rho(\mathrm{z}+\mathrm{b}/2) - \rho(\mathrm{z}-\mathrm{b}/2)}{\mathrm{r}_1{}^3 \mathrm{r}_2{}^3} \, \mathrm{d}\rho \, \mathrm{dz} \\
&= \frac{\mu_0 \mathrm{qm}}{4\pi} \int_0^\infty \int_0^\infty \frac{\rho^3 \ \mathrm{b} \ \mathrm{d}\rho \ \ \mathrm{dz}}{\sqrt{\rho^2+(\mathrm{z}+\mathrm{b}/2)^2}^{\,3} \ \sqrt{\rho^2+(\mathrm{z}-\mathrm{b}/2)^2}^{\,3}} \\
&= \frac{\mu_0 \mathrm{qm}}{4\pi}.
\end{aligned} \tag{3}$$

The above integral, though not found in integral tables, is easily shown to be unity by numerical integration. [Thomson structured the problem as we have and states that the result is easily obtained by integration, but he shows no details. Jackson, 1975, p 256, provides an analytical solution with a different structure for the integral.]

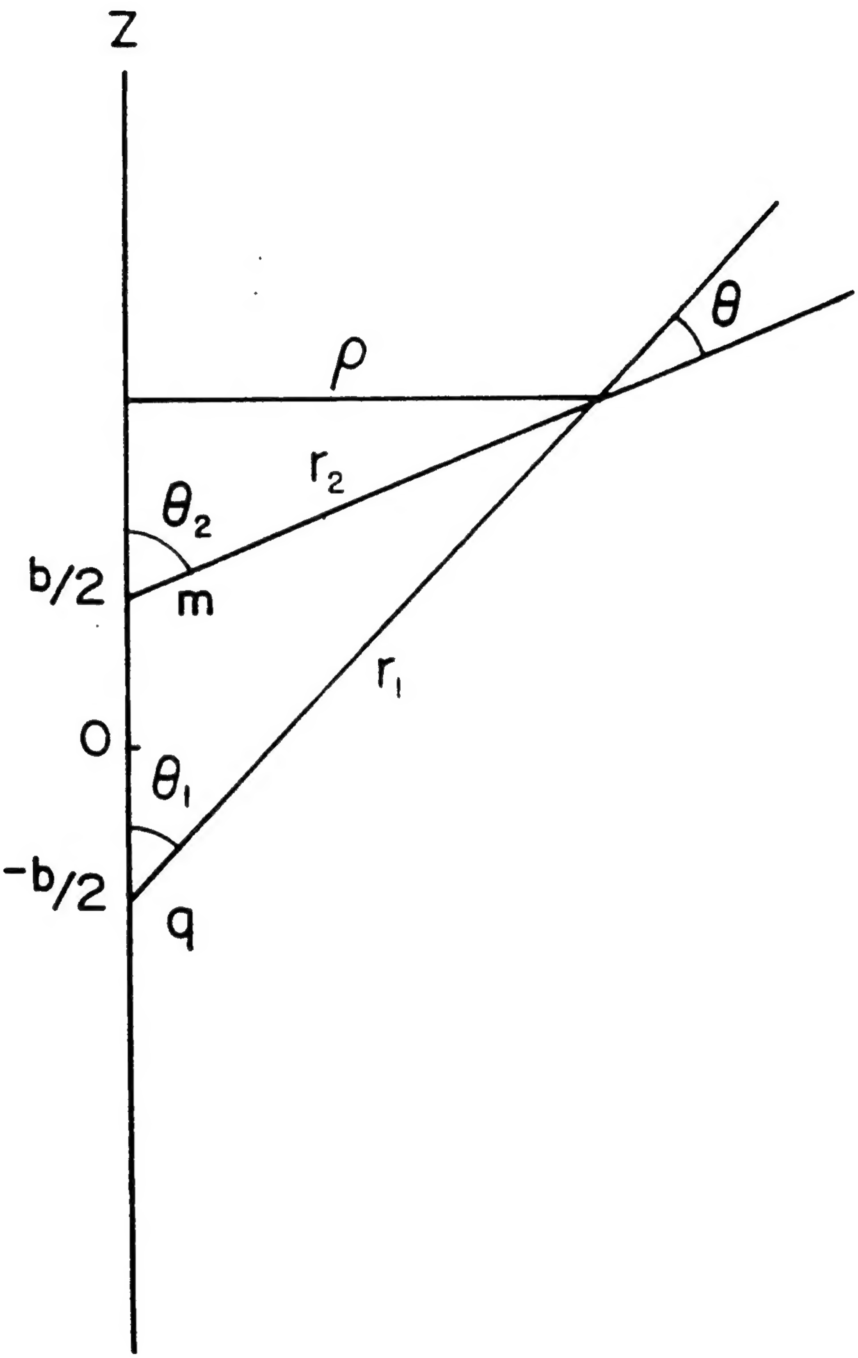


FIGURE 1

Pugh and Pugh [1967] have provided a very clear example in which use of the concept of emmd is essential in explaining the angular momentum of a system. Their system comprised a pair of concentric spheres, the inner one magnetized, with an electric field between them; as the system is charged, it develops mechanical angular momentum without the application of any external mechanical torques.

Thomson [1904b] was probably the first to provide a clear statement to the effect that the vector potential of a current distribution is equal to the electromagnetic momentum of a unit charge placed in the field of that current distribution. Nonetheless, the vector potential was long widely regarded as a purely mathematical convenience without physical meaning [Moullin, 1932; Aharonov and Bohm, 1959; Konopinski 1978, 1981]. Calkin [1966] and Konopinski [1978] rediscovered this property of the vector potential. However, even before their publications, it was common practice in quantum mechanics to regard the product of the vector potential and a charge q as part of the generalized momentum of the particle [Landau and Lifshitz, 1951; Aharonov and Bohm, 1959; Wangsness, 1963].

Identification of $\epsilon_0 \mathbf{E} \times \mathbf{B}$ as the emmd in vacuum helps make clear its physical nature because of its obvious relationship with the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, the rate of flow of elecctromagnetic energy per unit area. Owing to the equivalence of mass and energy (energy = mc^2), $\epsilon_0 \mathbf{E} \times \mathbf{B} = S/c^2$ is the rate of flow of mass per unit area, which is just momentum density. The

emmd in vacuum is simply the consequence of the flow of the mass equivalent of the energy. In a material medium, the nature of emmd is less obvious and sometimes complicated; it will be described later.

LONG SOLENOID AND AN ELECTRIC CHARGE

A useful example to consider in connection with electromagnetic momentum is a long solenoid with a charge q located a distance r from its axis, inside or outside the solenoid but otherwise near its midpoint. Let the radius of the solenoid be R and the magnetic induction within the solenoid be B. The magnetic field outside the solenoid is small and it is considered at first to be zero. The charge is the source of the relevant electric field, and the field of emmd associated with the charge can be evaluated inside the solenoid [Konopinski, 1981 p 160] even when the charge is outside the solenoid. The momentum associated with q is

$$\int_{\tau} \epsilon_0 \mathbf{E} \times \mathbf{B} \, d\tau = q \, \mathbf{A}, \tag{4}$$

where the integration is over the volume within the solenoid. **A** is the vector potential of the solenoid current at the point $\mathbf{r}_q$ where q is located, and

$$\mathbf{A}(\mathbf{r}_q) = \frac{\mu_0}{4\pi} \int_{\tau} \frac{\mathbf{J}(\mathbf{r})}{|\mathbf{r}_q - \mathbf{r}|} \, d\tau \, . \tag{5}$$

The electric field **E** is the field of the charge q, disregarding any perturbation due to conducting properties of the solenoid. This is illustrated in Figure 2a for r > R, along with the associated field of the Poynting vector **S**. The field

of the Poynting vector indicates an energy source on the left-hand side of the solenoid and an energy sink on the right-hand side, and the electromagnetic momentum describes the rate of flow of equivalent mass across the solenoid. Physically, the fields indicated in Figure 2a could be realized by making the solenoid from a large array of small current generators, each at the potential established by charge q. Then the current generators on the left-hand side would be sources of electromagnetic energy, and those on the right-hand side, sinks. This flow of energy across the solenoid is canceled by the effects of a charge distribution that is induced on the surface of the solenoid if it is a conductor. The charge distribution induced on the solenoid cancels the field of charge q inside the solenoid and modifies the field outside the solenoid so that it is perpendicular to the surface of the solenoid at the surface. The electric field of the induced charge distribution is illustrated in Figure 2b, along with its associated Poynting vector field; within the solenoid the fields exactly cancel those shown in Figure 2a.

The magnetic field outside the solenoid, though very small, is not zero, and it is important. It gives rise to fields of emmd outside the solenoid, and these are shown in Figures 2a and 2b by dotted lines. The fields external to the solenoid are very small compared to those inside, and they make only very small contributions to the flow of energy across the solenoid and to the total linear electromagnetic momenta associated with charge q and the induced charge distribution on the solenoid.

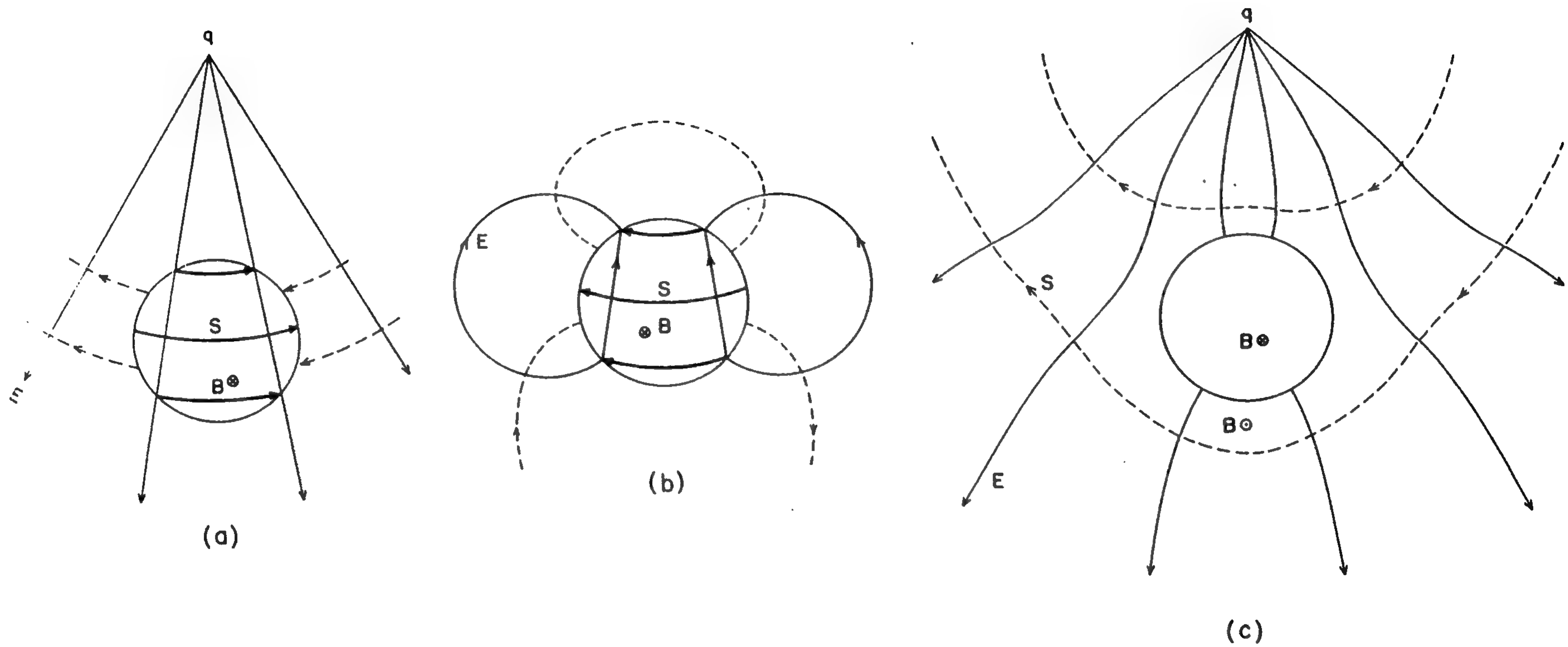


FIGURE 2

If the magnetic field is allowed to go to zero, q experiences an induced electric field to the right, the impulse delivered to it being

$$-(q/2\pi r)\int_{B=B}^{B=0}\frac{\partial}{\partial t}(\pi R^2B)\,dt = qBR^2/2r. \tag{6}$$

This is just the product of q and the vector potential $BR^2/2r$ at q, where r is the distance to q from the axis of the solenoid. It is also equal to

$$\int \epsilon_0 \mathbf{E}_q \times \mathbf{B}\,d\tau, \tag{7}$$

where $\mathbf{E}_q$ is the field of q and the integration is over the volume within the solenoid; the integral is evaluated in Appendix 1. Associated with the impulse to the right given to q, there is an equal impulse to the left given to the solenoid [Furry, 1969] as a consequence of the action of the induced electric field on the charge distribution induced on the solenoid by the charge q*. The force would be zero if there were no shielding charges; i.e., if the solenoid were a non–conductor. If a role is to be attributed to emmd inside the solenoid, it is that these impulses delivered to q and to the solenoid are due to the decay of their respective fields of emmd inside the solenoid, even though these fields totally cancel one another. It is clear that the electromagnetic momentum present in this picture is just angular momentum, as equal and opposite impulses act along lines separated by distance r. [Konopinski (1978; 1981) did not discuss the force on the solenoid, and Calkin (1966) incorrectly stated that it was zero. Calkin ignored the shielding charges; see Furry, 1969, footnote 16]. We will show shortly that the field of emmd that contains this angular momentum is external to the solenoid and that the cancelling fields within the solenoid simply reflect the properties of the external field.

*[An alternative approach to the evaluation of the force on the solenoid as B is reduced to zero has been provided by Costa de Beauregard (1967), but it is not conceptually correct. He states as a new law of electrodynamics that the force on a slowly varying current i in an electric potential field V is $\frac{di}{dt}\oint V\, dl/c^2$ and that the force on a varying magnetic dipole is $(1/c^2)\, \mathbf{E} \times d\mathbf{M}/dt$. Furry has shown that the force in question is actually that on the shielding charges induced on the solenoid or on the surface of a permanent magnet, and it is zero in their absence. However, with this reservation, Costa de Beauregard's relations are handy in evaluating the forces.]

Consider the development of the field of electromagnetic momentum as q is brought into position near an energized solenoid (rather than its being in place while the magnetic field is increased from zero). Bring q from infinity along the axis of the solenoid, considering this to be the z axis; in this way, it can be brought from infinity to the center of the solenoid without experiencing any magnetic force. Then move q along the y axis with velocity v so that it experiences a magnetic force $q\, \mathbf{v} \times \mathbf{B}$ to the left. This requires the application of an external force to the right in Figure 2a (where B is in the $-z$ direction) that will impart a total impulse qrB to q by the time it reaches position r, where $r < R$, or qRB when q reaches the boundary of the solenoid. An equal impulse, also to the right, is delivered to the solenoid due to its interaction with q, the magnetic field of the moving charge being the intermediary that transfers the impulse from q to the solenoid*. If this force on the solenoid is

resisted by an external force to prevent the solenoid from moving, the required impulse delivered to the solenoid by the external system is equal and opposite to that delivered to q. Thus angular momentum $qR^2B/2$ is delivered to the system by external forces as q moves from 0 to R. This angular momentum resides in electromagnetic form; if it is converted into mechanical form by letting the magnetic field go to zero at this point, the charge acquires an impulse $qRB/2$ to the right and the solenoid an impulse $qRB/2$ to the left.

*[That equal and opposite forces must by applied to the moving charge and the solenoid to maintain their specified trajectory and position has been shown by Furry [1969] in a more general treatment of the subject. He showed that, for a charge q and a solenoid m surrounded by a conducting shield s, the sum of the forces on the system is zero; the relevant forces being the $\mathbf{v} \times \mathbf{B}$ force $\mathbf{F}_q$ on charge q moving in the field of the solenoid, the $\mathbf{v} \times \mathbf{B}$ forces $\mathbf{F}_s$ on the shielding charges moving in the field of the solenoid, the $\mathbf{J} \times \mathbf{B}$ force $\mathbf{F}_{mq}$ on the solenoid due to the magnetic field of the moving charge q, and the $\mathbf{J} \times \mathbf{B}$ force $\mathbf{F}_{ms}$ on the solenoid due to the magnetic field of the moving shielding charges, where $\mathbf{J}$ is the magnetization current flowing round the solenoid. Although the sum of the forces is zero, there is not-pair wise cancellation. In our example with q approaching a long solenoid along the y axis from $+\infty$, both q and the shield receive small impulses (infinitesimally small as the solenoid becomes infinitely long) to the left due to forces $\mathbf{F}_q$ and $\mathbf{F}_s$. The largest impulses are to the solenoid due to $\mathbf{F}_{ms}$ acting to the left and $\mathbf{F}_{mq}$ acting to the right, their difference being small but to the right. The combined

impulse to the solenoid and the shield is to the right, equal and opposite to the impulse delivered to q by the electromagnetic interaction. Due to symmetry, the $\mathbf{v} \times \mathrm{B}$ force on q due to the magnetic field of the moving shielding charges and the $\mathbf{v} \times \mathrm{B}$ forces on the moving shielding charges due to the magnetic field of q do not enter into the current problem. Furry's treatment may be regarded as the basis for the reaction concept for the electromagnetic force exerted by one system upon another, that the forces are equal and opposite even if they are not in line. The reaction concept was also presented by V. H. Rumsey in lectures at the University of Illinois in 1956.]

As q is taken along the y axis beyond R, it experiences no magnetic force (in the approximation of negligible field outside the long solenoid) and its angular momentum $qR^2B/2$ must be conserved. Accordingly, the linear electromagnetic momentum associated with q decreases as $1/r$ and its value is $qR^2B/2r$, just the product of the vector potential $R^2B/2r$ and q. The solenoid has equal and opposite linear electromagnetic momentum. There is no alternative to the concept of emmd in explaining how the angular momentum exists in the system consisting of an energized solenoid and a charge q.

The above discussion has ignored the fact that B outside the solenoid, though small, is not zero. Taking the weak external magnetic field into account, there is a weak magnetic force on q as it moves along the y axis beyond R, and the force decreases very slowly with distance. The magnetic induction as a function of distance r along the y axis for a long solenoid of

length L and radius R (L >> R) is

$$B_e = 2\mu_0\alpha R^2/L^2\sqrt{1 + 4r^2/L^2}^3, \tag{8}$$

where α is the magnetizing current per unit length flowing round the solenoid. Near the solenoid $B_e = 2\mu_0\alpha R^2/L^2$, and it falls to about one–tenth of this value at $r = 0.95L$. The angular momentum conveyed to q in resisting this magnetic force as the particle goes to infinity along the y axis just cancels the angular momentum conveyed to the particle (and stored as emmd) as the particle proceeded from the axis of the solenoid to its surface, so the angular momentum at infinity is zero.

The above makes clear where the field of electromagnetic momentum exists for a charged particle q outside a long solenoid, and this is illustrated in Figure 2c, which shows schematically the electric field and the field of the Poynting vector. The field lines of emmd are roughly circular about q in the plane of the figure, with a detour around the solenoid. As there are no energy sources or sinks, the integral of the emmd over all space is zero; this can also be seen from the continuity of the field lines of the Poynting vector. Thus the total linear momentum of the system is zero, but the angular momentum is not. Although the fields outside the solenoid are weak, they are extensive, and the integrated angular momentum is finite. The volume within the solenoid makes no contribution to either the total linear or the total angular momentum. The angular momentum could be evaluated by integrating the product of the emmd and a radius vector over all space, but it is easier to make use of the property noted by Thomson, evaluating the angular

momentum in terms of the charge q and magnetic monopoles $\pi\alpha R^2$ and $-\pi\alpha R^2$ at the ends of the solenoid. Each pole in its interaction with charge q has angular momentum $\mu_0 q\pi\alpha R^2/4\pi = q\mu_0\alpha R^2/4$, with axial component $q\mu_0\alpha R^2/4\sqrt{1 + 4r^2/L^2}$. Therefore the angular momentum of the system is $q\mu_0\alpha R^2/2\sqrt{1 + 4r^2/L^2} \approx q\mu_0\alpha R^2/2$ for $r << L$. If the magnetic field is turned off with the charge and the solenoid held fixed, the linear momentum communicated to each of them is $q\mu_0\alpha R^2/2r\sqrt{1 + 4r^2/L^2}$, to the right in the case of the charge and to the left in the case of the solenoid. This transfer of momentum is due to the decay of the field of emmd, all of which is external to the solenoid. However, as we showed earlier, these impulses can be calculated from the cancelling fields inside the solenoid of the emmd for the charge q and for the induced charge distribution on the solenoid; the cancelling fields simply serve as proxies for the properties of the external field.

Much of what has been said in the context of $r > R$ is also applicable for $r < R$. The angular momentum possessed by the system for $r < R$ is $qBr^2/2$. The linear momentum associated with q, expressed in terms of the electromagnetic momentum within the solenoid, is $\epsilon_0\int \mathbf{E}_q \times \mathbf{B}\, d\tau$, where the integration is over the volume within the solenoid. The integration has been performed in Appendix 1, and for $r < R$, its value is $qBr/2$ in the $+x$ direction. The linear momentum associated with the solenoid is equal and opposite, i.e., $\epsilon_0\int \mathbf{E}_s \times \mathbf{B}\, d\tau = -\epsilon_0\int \mathbf{E}_q \times \mathbf{B}\, d\tau$, where $\mathbf{E}_s$ is the field of the charge distribution induced on the solenoid; this has magnitude $qBr/2$ and is in the $-x$ direction. Even though the fields of q and the charge distribution on

the solenoid have different topologies, the integrals of the Poynting vector over the volume inside the solenoid are equal and opposite.

INFINITESIMAL DIPOLE AND AN ELECTRIC CHARGE

The conversion of mechanical angular momentum into electromagnetic angular momentum and vice versa cannot be demonstrated with a magnetic monopole and a charge, but it can be with a dipole and a charge. The mechanical angular momentum that is conveyed to the system in bringing a charge from infinity into the proximity of a magnetic dipole exists in the form of electromagnetic angular momentum, and it can be released as mechanical angular momentum by letting the dipole decay to zero. The angular momentum is easily evaluated using Thomson's result for the angular momentum associated with a charge q and a magnetic monopole m. It immediately follows that the angular momentum associated with a charge q and a magnetic dipole M is

$$\mathbf{L} = -\frac{\mu_0 q M \sin\theta}{4\pi r}\,\hat{\boldsymbol{\theta}}, \tag{9}$$

where the charge is at r, θ relative to the dipole, θ being measured from the dipole axis.

To examine how angular momentum is put into the system as a charge is brought from infinity into the proximity of a magnetic dipole, any path can be selected; the one that we use here serves as an example. We consider a

dipole at the origin of our coordinate system with the z axis along the dipole axis. Let the coordinates of the charge be (x, 0, b) and let it move from $-\infty$ in the x direction with velocity v.

The electromagnetic angular momentum from Thomson's relation is

$$\mathbf{L}_{em} = -\frac{q\mu_0 M \sin\theta}{4\pi r}\hat{\theta} = \frac{q\mu_0 M}{4\pi b}(\hat{\mathbf{k}}\sin^2\theta\cos\theta + \hat{\mathrm{i}}\sin\theta\cos^2\theta). \tag{10}$$

The magnetic field in the xz plane has components:

$$B_x = -\frac{2\mu_0 M}{4\pi r^3}\sin\theta\cos\theta,\ \ B_y = 0, \text{ and }\ B_z = \frac{\mu_0 M}{4\pi r^3}(3\cos^2\theta - 1), \tag{11}$$

where $\theta = \tan^{-1}(-x/b)$. The magnetic force on q as it moves parallel to the x axis requires the application of an external force in the y direction to keep it on its prescribed path, and this external force is

$$\mathbf{F}_e = \frac{qv\mu_0 M}{4\pi r^3}(3\cos^2\theta - 1)\,\hat{\mathrm{j}}. \tag{12}$$

The force experienced by M is equal and opposite to that experienced by q, so another external force $-\mathbf{F}_e$ must be applied to M to keep it from moving. The two external forces constitute a couple that adds angular momentum to the system, the contribution during time interval dt being

$$d\mathbf{L}_{ef} = \frac{qv\mu_0 M}{4\pi r^2}(3\cos^2\theta - 1)\,\hat{\theta}\,dt. \tag{13}$$

As $v\,dt = dx = -r\,d\theta/\cos\theta$ and $r = b/\cos\theta$, this is

$$d\mathbf{L}_{ef} = -\frac{q\mu_0 M}{4\pi b}(3\cos^2\theta - 1)(-\hat{\mathbf{k}}\sin\theta - \hat{\mathrm{i}}\cos\theta)\,d\theta.$$

Then $\mathbf{L}_{ef} = \frac{q\mu_0 M}{4\pi b}\int_{\pi/2}^{\theta}(3\cos^2\theta - 1)(\hat{\mathbf{k}}\sin\theta + \hat{\mathrm{i}}\cos\theta)\,d\theta$

$$= \frac{q\mu_0 M}{4\pi b}[\cos\theta\sin^2\theta\,\hat{\mathbf{k}} + (\sin\theta\cos^2\theta - (1-\sin\theta))\hat{\mathrm{i}}]. \tag{14}$$

There is an additional torque that must be applied to the system as q moves along its prescribed path; the magnetic field of the moving charge produces a

torque on M, and an equal and opposite torque from an external source must be applied to M to stop it from turning. The magnetic field of q at M is $B_q = \frac{q\mu_0 vb}{4\pi r^3}\hat{j}$, so the torque exerted on M is $-\frac{q\mu_0 vbM}{4\pi r^3}\hat{i}$. The angular momentum conveyed to the system in resisting this torque is

$$\int_{-\infty}^{t} \frac{q\mu_0 vbM}{4\pi r^3}\, dt\, \hat{i} = -\frac{q\mu_0 M}{4\pi b}\int_{\pi/2}^{\theta} \cos\theta\, d\theta\, \hat{i} = \frac{q\mu_0 M}{4\pi b}(1-\sin\theta)\,\hat{i}. \tag{15}$$

Thus the total angular momentum delivered to the system is

$$L_e = \frac{q\mu_0 M}{4\pi b}[\hat{\mathbf{k}}\cos\theta\sin^2\theta + \hat{i}\sin\theta\cos^2\theta], \tag{16}$$

in agreement with the electromagnetic angular momentum of the system.

To release the electromagnetic angular momentum in mechanical form, let the dipole decay to zero. The impulse delivered to q can be evaluated by calculating the induced electric field. Consider the charge and the dipole to be fixed in position, thus requiring that impulses equal and opposite to those resulting from the decay of the dipole field be delivered from an external source. The flux of magnetic induction through a circle through q generated by revolution around the z axis is $(\mu_0 M/2r)\sin^2\theta$. The electric field at q is

$$\frac{1}{2\pi r\,\sin\theta}\frac{d}{dt}\left(\frac{\mu_0 M\,\sin^2\theta}{2r}\right),$$

and the integrated impulse delivered to q when the dipole is reduced to zero is

$$-\frac{q\mu_0 M\,\sin\theta}{4\pi r^2}\hat{j}.$$

The easiest way to calculate the impulse delivered to the dipole is to use Costa de Beauregard's [1967] relation for the force on a changing magnetic dipole in an electric field. The force is $\frac{E}{c^2}\times\frac{dM}{dt}$, and this yields $\frac{q\mu_0 M\,\sin\theta}{4\pi r^2}\hat{j}$

for the impulse, equal and opposite to the impulse delivered to the charge. Alternatively and conceptually better, this result can be asserted on the basis of Furry's conclusion that the force on the shielding charges on the magnet is equal and opposite to that on charge q. Thus the angular momentum released from the electromagnetic field as a consequence of the decay of M is $\frac{q\mu_0 M \sin\theta}{4\pi r}\hat{\theta}$, in agreement with both the electromagnetic angular momentum determined from Thomson's relation and the mechanical angular momentum added to the system by external forces as q was brought from infinity to position r,θ.

THE PARALLEL PLATE CAPACITOR

Further insight into the nature of electromagnetic momentum can be obtained by considering the charge and discharge of a parallel-plate capacitor in a magnetic field, the magnetic field being parallel to the plates. The example that we consider is illustrated in Figure 3. The parallel-plate capacitor is arranged so that moving a switch to position "a" charges the capacitor, and moving it to position "b" discharges it. A is the area of the plates, s is the distance between them, V is the potential difference supplied by the battery, and the electric field between the capacitor plates is $E = V/s$. The role of resistor R is to make the charging and discharging processes slow enough so that radiation can be ignored. The constant magnetic field B is parallel to the plates, directed into the plane of the figure. When the switch is

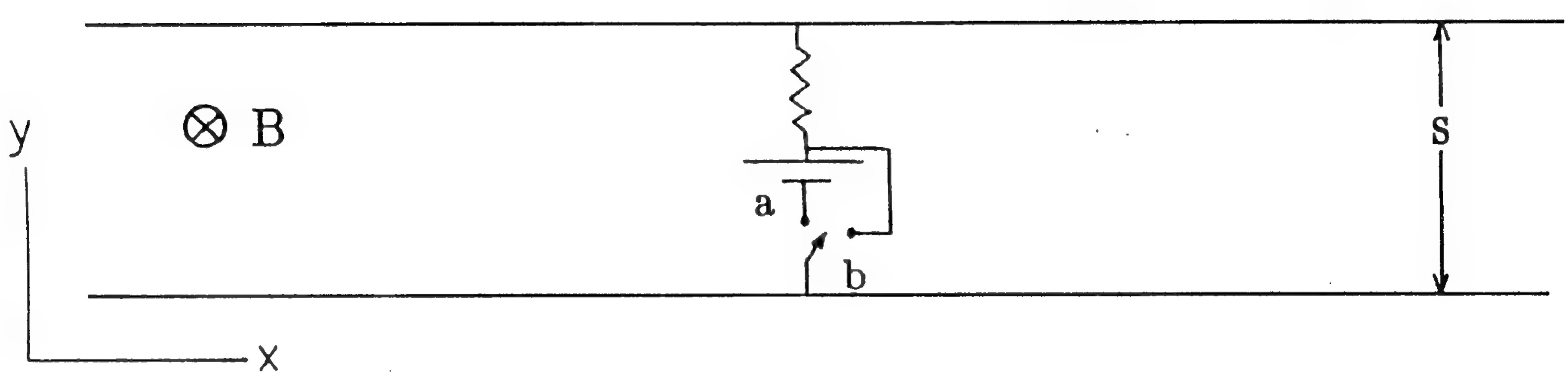


FIGURE 3

moved to position "a", a current I flows to charge the capacitor and a force to the left is exerted on the circuit joining the capacitor plates due to the interaction between the charging current and the magnetic field. The impulse delivered to the circuit is $-Bs\int I dt = -\epsilon_0 AsEB$, neglecting edge effects, where the positive direction is to the right.

Something must have acquired equal and opposite momentum, and this is the electromagnetic field within the volume of space between the capacitor plates; this volume has acquired emmd $\mathbf{g} = \epsilon_0\ \mathbf{E} \times \mathbf{B}$, which is directed towards the right. The total electromagnetic momentum between the plates is $\epsilon_0 AsEB$, equal and opposite to the impulse delivered to the circuit. Thus the mechanical system has acquired its momentum in association with the creation of an equal but opposite amount of electromagnetic momentum between the capacitor plates.

The example provides additional insight if the volume between the plates is filled by a dielectric of relative permittivity ϵ_r, in which case the charge acquired by the capacitor and the impulse delivered to the circuit are correspondingly larger by the factor ϵ_r. What is the emmd in this case? Practice varies as to how it should be defined, $\epsilon_0\ \mathbf{E} \times \mathbf{B}$ or $\mathbf{D} \times \mathbf{B} = \epsilon_r \epsilon_0\ \mathbf{E} \times \mathbf{B}$, the difference between the two being the momentum per unit volume delivered to the dielectric, described next. We consider the dielectric to be supported separately from the capacitor plates and the connecting circuit so that the effects of the forces acting on each can be considered separately.

Going back for the moment to the case with no dielectric present, consider the displacement current density $\mathbf{J}_d = \epsilon_0\, d\mathbf{E}/dt$ and its interaction with the magnetic field. The displacement current is directed downwards between the capacitor plates, and there is a $\mathbf{J}_d \times \mathbf{B}$ force per unit volume directed to the right and equal to $\epsilon_0\, d\mathbf{E}/dt \times \mathbf{B}$; the integral of this force over the time required to charge the capacitor is $\epsilon_0\, \mathbf{E} \times \mathbf{B}$. It is this impulse per unit volume directed towards the right that creates the emmd in vacuum in the space between the capacitor plates. Stated otherwise, the magnetic force on the displacement current creates emmd.

When the dielectric is in place, the displacement current includes a polarization term, and $\mathbf{J}_d = \epsilon_0\, d\mathbf{E}/dt + d\mathbf{P}/dt$. The first term provides the same impulse per unit volume that existed without the dielectric; it creates pure field (or vacuum) momentum. The second term involves an impulse per unit volume delivered to the dielectric as a consequence of the polarization current. It is this momentum delivered to the dielectric that is considered to be part of the emmd when it is defined as $\mathbf{D} \times \mathbf{B}$. Further, the example provides an argument that it more appropriate from a physical point of view to consider the emmd to be given by $\mathbf{D} \times \mathbf{B}$ rather than by $\epsilon_0\, \mathbf{E} \times \mathbf{B}$, with the mechanical momentum treated separately in the latter case. During the discharge of the capacitor, the emmd collapses and the circuit receives an impulse to the right $\epsilon_r \epsilon_0 EB$ and the dielectric an impulse to the left $(\epsilon_r - 1)\epsilon_0 EB$, irrespective of the momentum state of the dielectric when the discharge commences. During the charging of the capacitor, the dielectric did

receive an impulse to the right. Whether the dielectric retains this momentum or delivers it to something else (the dielectric support structure, for example), an impulse to the left is delivered to the dielectric when the capacitor is discharged. [If the dielectric in the capacitor example is fixed to the capacitor plates, then the impulse delivered to the capacitor–dielectric combination is just the change in $\epsilon_0 \mathbf{E} \times \mathbf{B}$, the same impulse that would occcur without the dielectric in place. This interesting fact should not divert attention from the real nature of the interactions.] This argues that the emmd, including its interaction with the dielectric, should be regarded as as a property of the electromagnetic field and equal to $\mathbf{D} \times \mathbf{B}$; it is not simply a matter of momentum being stored in the dielectric while the capacitor is charged. It is a property of the electromagnetic field that an impulse be delivered to the dielectric whenever there is a change in its polarization, the impulse being $(\epsilon_r - 1)$ times the change in $\epsilon_0 \mathbf{E} \times \mathbf{B}$.

The emmd between the capacitor plates in this example can be produced by a different sequence of events. Suppose that the capacitor without the dielectric is charged in the absence of any magnetic field, and that the magnetic field is then applied slowly with $\partial B/\partial t$ constant until it reaches a specified value B. While the magnetic field is increasing, there is an induced electric field described by $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. This gives rise to a force to the left on the charges on the capacitor plates, the total force in the +x direction being $-Qs\ \partial B/\partial t = -\epsilon_0 AsE\ \partial B/\partial t$, where $Q = \epsilon_0 AE$ is the charge on the capacitor plates. The force on the capacitor plates can also be seen as the

force of reaction associated with the pressure gradient force in the electric field. The pressure gradient force in the electric field is $-\frac{\partial}{\partial x}\frac{\epsilon_0 E^2}{2}$; this force per unit volume to the right is equal to $-\epsilon_0 E\frac{\partial E}{\partial x} = \epsilon_0 E\frac{\partial B}{\partial t}$ (since $\nabla \times E = \frac{\partial E}{\partial x}$ in this example). This pressure gradient force produces the emmd. The force of reaction must be borne by the capacitor plates. Thus the force on the capacitor plates associated with the growth of emmd between the capacitor plates is $-\epsilon_0 AsE\frac{\partial B}{\partial t}$. The total impulse delivered to the plates when the magnetic field has reached the value B is $-\epsilon_0 AsEB$; this is associated with the generation of emmd between the capacitor plates. Accordingly, the emmd between the capacitor plates is $\epsilon_0 EB$, directed to the right as before, but this time resulting from the time rate of change of B rather than of E. Analogous to the magnetic force $J_d \times B$ on the displacement current J_d, which produces a pressure gradient in the magnetic field that in turn creates emmd, the changing magnetic field produces a pressure gradient in the electric field that creates emmd.

If this sequence is repeated with the dielectric in place, the impulse delivered to the plates is larger by the factor ϵ_r, and the increase is associated with an impulse to the right in the amount $(\epsilon_r - 1)\epsilon_0 AsEB$ delivered to the dielectric. The impulse to the dielectric is due to the action of the induced electric field on the bound charge $(\epsilon_r - 1)\epsilon_0 AE$ on the surfaces of the dielectric (or on the surfaces of volume elements of the dielectric), the charge being negative on the upper surface of the dielectric slab (or of the volume elements). It can also be seen in terms of the negative gradient of the electric

field energy density $-\frac{\partial}{\partial x}\frac{\epsilon_r\epsilon_0 E^2}{2}$, which is larger than the pure-field value by the factor ϵ_r. The force per unit volume on the dielectric is $(\epsilon_r - 1)\epsilon_0 E\ \partial B/\partial t$, and the impulse per unit volume that has been delivered to the dielectric when the magnetic field reaches the value B is $(\epsilon_r - 1)\epsilon_0 EB$. As this should be considered to be part of the emmd, the total emmd is $\epsilon_r\epsilon_0 EB = DB$. [Lorrain (1980) has described forces on the medium which reduce for the simple system considered here to $\mu_0(\partial \mathbf{P}/\partial t) \times \mathbf{H}$ and $\mu_0 \mathbf{P} \times (\partial \mathrm{H}/\partial t)$, which are equivalent to the expressions used here.]

If B and E increase in proportion, the contributions to emmd from $\partial E/\partial t$ and $\partial B/\partial t$ are equal, but the nature of the forces exerted on the medium by the two terms is different. The force associated with the variation in E is due to the polarization current interacting with the magnetic field, thus conveying momentum to the medium; this is also reflected in the spatial distribution of B as perturbed by the polarization current and in the gradient of magnetic field energy density. The force associated with the time variation of B is due to an electrical force on the polarization surface charges of volume elements of the dielectric, and this is reflected in the spatial distribution of E and the gradient of the electric field energy density; the force per unit volume on the dielectric is the negative gradient of the electric field energy density $\epsilon_r\epsilon_0 E^2/2$ less the negative gradient of the pure electric field energy density $\epsilon_0 E^2/2$, or $(\epsilon_r - 1)\epsilon_0 E^2/2$. As this force per unit volume due to the time variation in B is not easily visualized, it might be overlooked, which would

lead to a factor of two error in evaluating the mechanical momentum delivered to the medium.

The above discussion has not considered edge effects and fields outside the volume between the capacitor plates. They can be seen to be inconsequential by considering the equivalent of a guard ring. Visualize the capacitor as being simply an element of a much larger capacitor consisting of closely spaced coaxial spheres or coaxial cylinders with the outer surface grounded. [If the source of the magnetic field is considered to be a large solenoid with the capacitor along its axis, these edge effects and the interactions between the charging current and shielding currents induced on the surface of the solenoid are important, and it makes an interesting exercise to discuss them.]

RADIATION PRESSURE

There are several ways to evaluate the radiation pressure of electromagnetic waves – the photon flux times the momentum of the photon, the Maxwell stress tensor, and the flux of electromagnetic momentum. In addition, consideration of conservation of energy and momentum provides clear and unambiguous constraints that are useful in interpreting the results obtained from the Maxwell stress tensor and the emmd. We will consider the radiation pressure on idealized non-reflecting surfaces in idealized non-absorbing media: linear isotropic non-absorbing dielectrics characterized by a

single oscillator whose natural frequency ω_0 is greater than the wave frequency ω, and non-absorbing unmagnetized cold plasmas whose plasma frequency ω_p is less than the wave frequency. [Following the practice of Nayfeh and Brussel [1985], we characterize these as simple media.]

Empty Space

Consider a beam of irradiance I or photon flux density $I/h\nu$ propagating in the +z direction. In vacuum the photon momentum is $h/\lambda_0 = \hbar k_0$, where λ_0 and k_0 are the wavelength and radian wave number in vacuum. The flux of momentum per unit area carried by photons is $(I/h\nu)\, h/\lambda_0 = I/c$, where c is the velocity of light, and this is the radiation pressure on a surface that absorbs the beam. Regarded as an electromagnetic wave described by $E = E_0\cos(\omega t - k_0 z)$ and $\omega = 2\pi\nu$, the amplitude of the electric field is $E_0 = \sqrt{2I/c\epsilon_0}$. The Maxwell stress tensor indicates an average pressure $\epsilon_0 E_0^2/4 + B_0^2/4\mu_0 = \epsilon_0 E_0^2/2 = I/c$. The emmd is $\epsilon_0 EB = \epsilon_0 E^2/c$, and its average value is $\epsilon_0 E_0^2/2c = I/c^2$. Its flow velocity is c, so the flux of momentum per unit area is $\epsilon_0 E_0^2/2 = I/c$.

Considering instantaneous values, the electromagnetic stress per unit volume indicated by the Maxwell stress tensor is

$$-\frac{\partial}{\partial z}\, \epsilon_0 E_0^2 \cos^2\varphi = -\, k_0 \epsilon_0 E_0^2\, 2 \cos\varphi \sin\varphi = -\, k_0 \epsilon_0 E_0^2 \sin 2\varphi, \tag{17}$$

where $\varphi = \omega t - kz$. This produces emmd in the +z direction

$$\int -\, k_0 \epsilon_0 E_0^2 \sin 2\varphi \, dt = \frac{k_0}{2\omega}\, \epsilon_0 E_0^2 \cos 2\varphi + c_1 = \frac{1}{c}\, \epsilon_0 E_0^2 \cos^2\varphi, \tag{18}$$

where c_1 has been chosen on the basis that the emmd is zero when E and B are

zero. As $B = E/c$ for electromagnetic waves in empty space, this is just equal to the emmd, making it clear that the electromagnetic stress per unit volume produces the emmd that propagates with the waves.

Dielectrics

Next consider a beam of the same irradiance and frequency in an idealized normally dispersive, non-absorbing dielectric characterized by relative permittivity ϵ_r and index of refraction $n = \sqrt{\epsilon_r}$. In the dielectric, the photon momentum is $h/\lambda = \hbar k = nh/\lambda_o$, or n times the momentum in vacuum, where λ and k are the wavelength and radian wave number in the dielectric. That the momentum of a photon in a medium of refractive index n is $h/\lambda = nh/\lambda_o$ has been established within about 0.1% by measurements made by Jones and Leslie [1978]. The flux density of momentum carried by photons is $(I/h\nu)h/\lambda = I/v_p$, where v_p is the phase velocity in the dielectric. The photon momentum increases by the factor n as it enters the dielectric, and an outward force must be exerted on the entry surface of the dielectric associated with this change. If a beam enters the dielectric from space without any reflective loss (so that I is the same in space and in the dielectric), the magnitude of the outward force is $(n-1)I/c$.

Now consider the beam in terms of an electromagnetic wave of angular frequency ω propagating in the z direction in a dielectric characterized by a single resonant frequency ω_o, where $\omega < \omega_o = \sqrt{\kappa/m}$, κ being the force constant for the restraining forces on the electrons. Only a very brief summary of the

mathematical properties of these waves will be given here. [For a more complete description of these waves using the same notation used here, see Johnson, 1990, Appendix B. For a more general discussion of electromagnetic waves in simple absorbing dielectrics, see Loudon, 1970.] The electric field is $E = E_0 \cos\varphi$, where E_0 is the amplitude and $\varphi = \omega t - kz$ is the phase. The magnetic field is $B = B_0 \cos\varphi$, where $B_0 = E_0/v_p$ and $v_p = \omega/k$ is the phase velocity. The flow of energy per unit area in the direction of propagation, given by the Poynting flux, is $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, and its average value is the irradiance

$$I = \left(\frac{\epsilon_r \epsilon_0 E_0{}^2}{4} + \frac{B_0{}^2}{4\mu_0}\right) v_p = \frac{\epsilon_r \epsilon_0 E_0{}^2}{2} v_p. \tag{19}$$

In addition to the energy density $\frac{\epsilon_r \epsilon_0 E^2}{2}$ flowing at the phase velocity, there is non–propagating energy in the form of kinetic energy of the electrons and potential energy in the force field that determines the equilibrium positions of the electrons. [Visscher (1988) very aptly uses the term elastic potential energy to identify this potential energy. Brillouin (1960) calls it potential energy, and we have followed his usage.] The electrons oscillate in response to the electric field with velocities

$$v_e = \frac{\omega e E_0 \sin\varphi}{m(\omega_0{}^2 - \omega^2)} \tag{20}$$

and kinetic energies

$$m v_e{}^2/2 = \frac{\omega^2 \omega_p{}^2}{N(\omega_0{}^2 - \omega^2)^2} \frac{\epsilon_0 E_0{}^2}{2} \sin^2\varphi, \tag{21}$$

where $\omega_p{}^2 = Ne^2/\epsilon_0 m$.

Consider what happens during two successive quarter-cycles starting at a point where the electrons pass through their equilibrium positions. At the

beginning of the first quarter cycle, the electric field energy and the potential energy are both zero and the kinetic energy has its maximum value. During the first quarter cycle, the kinetic energy falls to zero and in so doing produces potential energy. However, over this same time interval electrical energy is expended to force the electrons to oscillate at a frequency less than their natural frequency, and this causes still more potential energy to be produced, the additional amount per unit volume being $\chi_e \epsilon_0 E_0^2/2$, where χ_e is the electrical susceptibility. During the next quarter cycle, all of the potential energy disappears. Part of it reappears as kinetic energy, which has its maximum value again at the end of the second quarter cycle. The remaining part acts just as if it were part of the propagating electric field energy; it is considered to be electric field energy, and it is taken into account through the use of a relative permittivity ϵ_r that is greater than unity.

The kinetic energy density and that part of the potential energy density produced from kinetic energy has a constant sum

$$\frac{\omega^2 \omega_p{}^2}{(\omega_0{}^2 - \omega^2)^2} \frac{\epsilon_0 E_0{}^2}{2} = \frac{\omega^2 \chi_e}{\omega_0{}^2 - \omega^2} \frac{\epsilon_0 E_0{}^2}{2} = \left(\frac{v_p}{v_g} - 1\right) \epsilon_r \frac{\epsilon_0 E_0{}^2}{2}. \qquad (22)$$

This constitutes the non–propagating energy density associated with the wave field in a simple dielectric, and it is entirely mechanical in nature (potential and kinetic); it has no electromagnetic component, and it does not propagate with the waves. Nevertheless, it must be present in the right amount and in synchronism with the waves in order for the electromagnetic waves to exist in the dielectric. If the waves increase in amplitude with time, the non–propagating energy must increase at the expense of electromagnetic energy, and if

they decrease, non–propagating energy must disappear by conversion to propagating electromagnetic wave energy. [We do not discuss the transient involved in this change; it is discussed in a different context by Summerfeld (1914), Brillouin (1914;1960), and Stratton (1941).]

The non–propagating energy density satisfies the energy conservation requirement that the total energy associated with a wave packet must equal the energy transported by the waves across a plane perpendicular to the direction of propagation as the wave packet passes by. The propagating energy in a wave packet at any instant is less by the factor v_g/v_p than the total energy of the wave packet; the non–propagating energy density makes up the difference. The energy conservation implied by the statement can be expressed $v_g\langle W_t\rangle = v_p\langle W_p\rangle$, where v_g and v_p are the group and phase velocities, $\langle W_t\rangle$ is the average total energy density associated with the wave field, and $\langle W_p\rangle$ is the average propagating energy density. In dielectrics, $\langle W_p\rangle = \langle W_{em}\rangle$, the sum of the average electric and magnetic energy densities $\epsilon_r\epsilon_0 E_0^2/4$ and $B_0^2/4\mu_0$.

The group velocity is the weighted average velocity of all the energy associated with the wave field (propagating and non-propagating), and as indicated above it can be determined from the properties of a wave field of constant amplitude, although it is more conventional to express it in terms of interference between waves of slightly differing frequencies. All of the energy propagation occurs by means of electromagnetic waves and at the phase

velocity, the instantaneous value being given by the Poynting vector, which is equal to the product of the propagating energy density and the phase velocity. Instantaneous energy flow at the group velocity is meaningless; instantaneoous flow can be expressed only in terms of the phase velocity. The eventual fate of the non-propagating energy as the wave field decays is conversion to propagating electromagnetic energy; thus it eventually contributes to the energy flow and to radiation pressure, but not while it is in the form of kinetic and potential energy. As a wave packet moves through a dielectric medium, the individual waves propagate at a velocity greater than that of the group, and the waves move forward in the group. In the forward half of the group, the individual waves diminish in amplitude as they deposit energy in the medium, giving up energy to the non-propagating kinetic and potential energy of the dielectric. In the rear half of the group, the individual waves grow in amplitude at the expense of the non-propagating energy as they move forward in the group.

Something very similar must occur for momentum. In the forward half of the group, the decaying waves give up momentum as well as energy to the medium. In the rear half of the group where new waves are launched, the growing waves deliver recoil momentum to the medium, directed opposite to the direction of propagation. It is convenient to regard the momentum as being stored in the medium during the passage of the wave packet, although, just as in the capacitor example discussed earlier, it is not actually necessary for the momentum to be retained by the medium while the wave field is

present. The total momentum in a wave packet, including the momentum that will be extracted from the medium when the wave field decays, is thus equal to the propagating momentum carried past a given point by waves at the phase velocity as the wave packet passes by. Thus conservation of momentum is satisfied.

The emmd is $\epsilon_r\epsilon_o EB = \epsilon_r\epsilon_o E^2/v_p$, and its average value is $\epsilon_r\epsilon_o E_o{}^2/2v_p = I/v_p{}^2$. The flow velocity of the emmd is v_p, therefore the average flux of momentum per unit area is $\epsilon_r\epsilon_o E_o{}^2/2 = I/v_p$, in agreement with the flux of momentum carried by photons. Half of the emmd is produced by $\partial B/\partial t$ and half by $\partial E/\partial t$. Thus an amount $\epsilon_r\epsilon_o E^2/2v_p = (1 + \chi_e)\epsilon_o E^2/2v_p$ relates to $\partial E/\partial t$; of this, $\epsilon_o E^2/2v_p$ is pure-field momentum and $\chi_e\epsilon_o E^2/2v_p$ is mechanical momentum produced by the polarization current. These quantities can be related to energy densities, $\epsilon_o E^2/2$, the pure electric field energy density, and $\chi_e\epsilon_o E^2/2$, that part of the potential energy density that acts as if it were electric field energy and propagates with the waves. The other half of the emmd, also equal to $(1 + \chi_e)\epsilon_o E^2/2v_p$, relates to $\partial B/\partial t$; it corresponds to the magnetic field energy density $B^2/2\mu_o = (1 + \chi_e)\epsilon_o E^2/2$ divided by v_p. This half of the emmd also consists of $\epsilon_o E^2/2v_p$ pure field momentum and $\chi_e E^2/2v_p$ mechanical momentum. Thus the magnetic field energy density has associated with it the mechanical momentum that is produced by the induced electric field force acting on the charges in the polarized dielectric, and this mechanical momentum propagates with the waves.

The non-propagating momentum associated with the waves is

$$\left(\frac{v_p}{v_g} - 1\right) \frac{\epsilon_r \epsilon_0 E_0^2}{2v_p} = \frac{\omega^2 \omega_p^{\,2}}{(\omega_0^{\,2} - \omega^2)^2} \frac{\epsilon_0 E_0^{\,2}}{2v_p} = \frac{\omega^2 \chi_e}{\omega_0^{\,2} - \omega^2} \frac{\epsilon_0 E_0^{\,2}}{2v_p}; \tag{23}$$

this is the amount of mechanical momentum that was deposited in the medium when the wave field was established and that will be extracted from it when the wave field decays. When this momentum density is added to the average propagating momentum density (i.e., the emmd), the total average momentum density associated with the wave field is

$$\frac{v_p}{v_g} \frac{\epsilon_r \epsilon_0 E_0^2}{2v_p} = \frac{\epsilon_r \epsilon_0 E_0^2}{2v_g}, \tag{24}$$

which obviously satisfies the requirement for conservation of momentum. [Jones (1978) by rather similar reasoning concluded that a fraction of the momentum of a photon is tangibly in the medium, and he classified this tangible component as mechanical.]

If the emmd is expressed as $\epsilon_0 \mathbf{E} \times \mathbf{B}$ rather than $\epsilon_r \epsilon_0 \mathbf{E} \times \mathbf{B}$, then the propagating mechanical momentum of the dielectric must be taken into account explicitly in order to obtain the total propagating momentum density. Use of vacuum or the Abraham stress tensor demands that the photon beam be accompanied by a mechanical force density (the Abraham force) transported by the medium, just sufficient to result in the observed radiation pressure [Ratcliff and Peak, 1972; Jones, 1978; Lorrain, 1980]. [Kranys (1979) states that the acceptance of a separate force field makes it impossible to discriminate between the macroscopic media or Minkowski and the Abraham forms of the stress tensor on the basis of experiment.] However, this assumption of a separate propagating mechanical force field is conceptually

unacceptable for a collection of uncoupled oscillators, the usual approximation for a simple dielectric. The momentum is carried forward by electromagnetic wave propagation, not by mechanical means.

The problem associated with the concept of propagating mechanical momentum by other than electromagnetic wave propagation is not encountered with the macroscopic media form of the stress tensor, where the mechanical momentum that propagates with the waves is included as part of the electromagnetic stress. This indicates an average pressure on an idealized absorbing surface of $\epsilon_r\epsilon_0 E_0{}^2/4 + B_0{}^2/4\mu_0 = \epsilon_r\epsilon_0 E_0{}^2/2 = I/v_p$, in agreement with the flux of momentum carried by photons. The stress per unit volume is

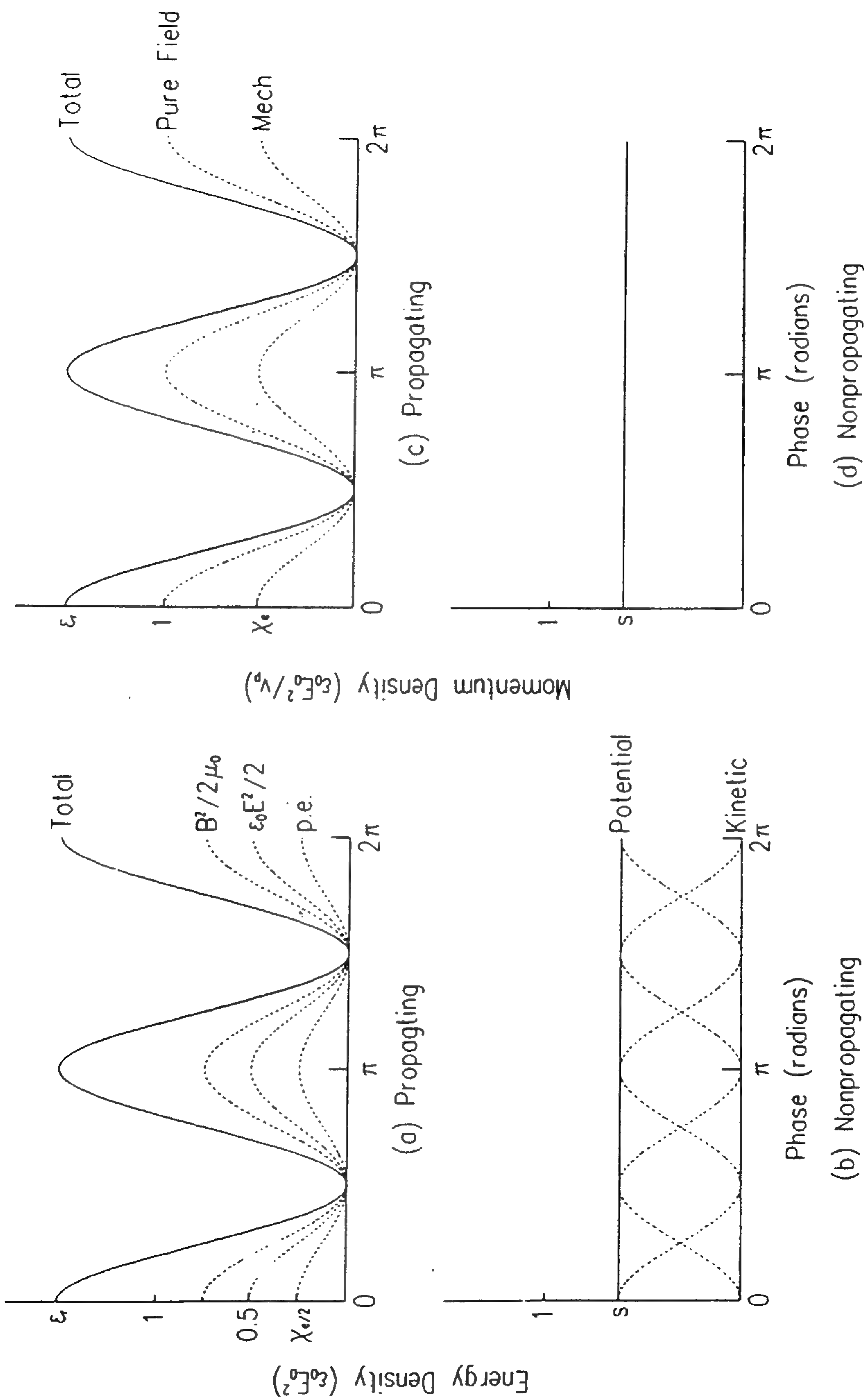
$$-\frac{\partial}{\partial z}\,\epsilon_r\epsilon_0 E_0{}^2\cos^2\varphi = -\,k\epsilon_r\epsilon_0 E_0{}^2\,2\cos\varphi\sin\varphi = -\,k\epsilon_r\epsilon_0 E_0{}^2\sin 2\varphi. \tag{25}$$

This produces momentum density in the z direction $\epsilon_r\epsilon_0 E_0{}^2\cos^2\varphi/v_p$; of this, $\epsilon_0 E_0{}^2\cos^2\varphi/v_p$ is pure field momentum and $\chi_e\epsilon_0 E_0{}^2\cos^2\varphi/v_p$ is mechanical momentum that acts as if it were part of the emmd. Half of the latter is produced by the magnetic force on the polarization current and half by the electrical force on the polarization charges.

The above relationships are set forth in Table 1 and illustrated in Figure 4. The fraction of the average total momentum that is mechanical in origin can be readily evaluated from Table 1; it is

$$\left(\frac{v_p}{v_g} - 1 + \frac{\chi_e}{\epsilon_r}\right)\frac{v_g}{v_p} = 1 - 1/nn_g, \tag{26}$$

where n_g is the group index of refraction. This was first noted by Arnaud



Energy Density ($\varepsilon_0 E_0^2$)
ε_t
1
0.5
$\chi_{e/2}$
Total
$B^2/2\mu_0$
$\varepsilon_0 E^2/2$
p.e.
0
π
2π
(a) Propagting
s
1
Potential
Kinetic
Phase (radians)
(b) Nonpropagating
Momentum Density ($\varepsilon_0 E_0^2/v_p$)
ε_t
1
χ_e
Total
Pure Field
Mech
(c) Propagating
Phase (radians)
(d) Nonpropagating

FIGURE 4

[1976], Player [1975], Rogers [1975], and Jones [1975; 1978]. It has the interesting and important property of being zero for $nn_g = 1$ or $v_p v_g = c^2$.

Plasmas

Consider electromagnetic waves with frequency ω greater than the plasma frequency ω_p in an idealized cold plasma. [See Johnson, 1990, Appendix C for a more complete mathematical description of these waves using the same notation used here. See Booker, 1984, for a much more general treatment of electromagnetic waves in cold plasmas, using notation quite similar to that used here.] The plasma is characterized by an index of refraction n that is less than unity. Use of the relationship $p = h/\lambda$ for the photon momentum yields a value that is less than that in free space. This indicates a radiation pressure on an absorbing surface in the plasma equal to $(I/h\nu)nh/\lambda_o = nI/c = I/v_p$, which is less than would exist in the absence of the plasma.

The Poynting flux in the plasma is $\epsilon_r{}' \epsilon_0 E^2 v_p$, with

$$\epsilon_r{}' = 1 - \omega_p{}^2/\omega^2 = 1 + \chi_e{}' = n^2 < 1. \tag{28}$$

In evaluating the Poynting flux, $\epsilon_r{}'$ plays a role similar to that of the relative permittivity ϵ_r in a dielectric, and $\epsilon_r{}'$ is sometimes referred to as a relative permittivity or dielectric constant. Booker [1984], for example, says that it is convenient to call $\epsilon_r{}'$ a dielectric constant. However, plasmas are not dielectrics, and care must be exercised in attributing dielectric properties to them. They do not share the usual property of simple media that the electric

and magnetic field energy densities for electromagnetic waves are equal to one another, and the electric field energy density is $\epsilon_0 E^2/2$, not DE/2.

For electromagnetic waves in plasmas, the pure electric field energy density is greater than the magnetic. The electric field energy density is $\epsilon_0 E^2/2$ and the magnetic field energy density is $B^2/2\mu_0 = DE/2 = \epsilon_r' \epsilon_0 E^2/2$. Of the total electric field energy density $\epsilon_0 E^2/2$, only the fraction ϵ_r' is associated with the propagating waves, while the remainder,

$$\frac{1 - \epsilon_r'}{2} \epsilon_0 E^2 = -\chi_e' \epsilon_0 E^2/2 = \frac{\omega_p^2}{\omega^2} \epsilon_0 E^2/2, \tag{27}$$

is associated with plasma oscillations and does not propagate; this part of the electric field energy density is produced by the expenditure of the kinetic energy of the oscillating electrons, the kinetic energy being $\frac{1 - \epsilon_r'}{2} \frac{\epsilon_0 E_0^2}{2} \sin^2\varphi$. The weighted average flow velocity of all the energy density, comprising the propagating and non-propagating electric field energy, the magnetic field energy, and the kinetic energy of the oscillating electrons, is the group velocity. The non-propagating part of the electric field energy density exchanges cyclically with the kinetic energy. It involves charge separation in the plasma that acts to increase the electric field, and one might expect this to contribute to electric field stress in the same way as charge applied to a parallel-plate capacitor, the only difference being that the energy source is the kinetic energy of the electrons rather than external. This suggests that the electric field energy density in excess of $\epsilon_r' \epsilon_0 E^2/2$ would contribute accordingly to the radiation pressure. However, the stress due to the non-propagating electric field energy density acts on the plasma and it does not

propagate with the waves, nor does it contribute to the radiation pressure. This will be discussed shortly.

The total energy density can be expressed either as the sum of the propagating energy density $\epsilon_r' \epsilon_0 E^2$ (all electromagnetic, but with a mechanical component) and the non-propagating energy density $\frac{1 - \epsilon_r'}{2} \epsilon_0 E_0^2$ (on average, half electric and half kinetic), or as the sum of the electric field energy density $\epsilon_0 E^2/2$, the magnetic field energy density $\epsilon_r' \epsilon_0 E^2/2$, and the kinetic energy density, $\frac{1 - \epsilon_r'}{2} \epsilon_0 E_0^2 \sin^2\varphi$. The average value of the total energy is $\epsilon_0 E_0^2/2$.

As the flow of energy indicated by the Poynting vector is $v_p \epsilon_r' \epsilon_0 E^2$, the flow of momentum is $\epsilon_r' \epsilon_0 E^2$, and the propagating momentum density is $\frac{1}{v_p} \epsilon_r' \epsilon_0 E^2$. We cannot immediately classify this momentum density as being pure field momentum or partly mechanical. The non–propagating energy density is $\frac{1 - \epsilon_r'}{2} \epsilon_0 E_0^2$, constant with time, but made up of two components – the non-propagating electric field energy density $\frac{1 - \epsilon_r'}{2} \epsilon_0 E_0^2 \cos^2\varphi$ and the kinetic energy density $\frac{1 - \epsilon_r'}{2} \epsilon_0 E_0^2 \sin^2\varphi$. This energy flows cyclically back and forth between the two forms as the plasma oscillates. The corresponding non-propagating momentum density is $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2$. We anticipate that this is also made up of two terms, $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2 \cos^2\varphi$ and $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2 \sin^2\varphi$. We can identify the first of these as non-propagating pure field momentum, as it corresponds to the non-propagating electric field energy density divided by v_p. The second can be identified as the mechanical momentum imparted to the

oscillating electrons by the $-e\ \mathbf{v} \times \mathbf{B}$ forces; it is equal to the kinetic energy divided by v_p. The momentum flows cyclically back and forth between these two forms as the plasma oscillates, in unison with the back and forth flow between kinetic energy and non-propagating electric field energy. The total average momentum density associated with the wave field in the plasma is the sum of the non-propagating momentum density plus the average propagating momentum density, or

$$\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0{}^2 + \frac{\epsilon_r' \epsilon_0 E_0{}^2}{2v_p} = \frac{\epsilon_0 E_0{}^2}{2v_p}. \tag{29}$$

As $v_g v_p = c^2$ and $v_g/v_p = \epsilon_r'$, the total average momentum density is $\frac{v_p}{v_g} \frac{\epsilon_r' \epsilon_0 E_0{}^2}{2v_p}$. This obviously satisfies the conservation requirement, as it is $\frac{v_p}{v_g}$ times the propagating momentum density.

The propagating momentum density is

$$\frac{1}{v_p} \epsilon_r' \epsilon_0 E^2 = \frac{1}{v_p}(1 + \chi_e'/2 + \chi_e'/2)\epsilon_0 E^2. \tag{30}$$

We can identify a part of the propagating momentum density, $\frac{1}{v_p} \frac{\chi_e'}{2} \epsilon_0 E^2 = -\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E^2$, as mechanical in nature, arising from the electrical forces on the polarization charges, but negative because the polarization is negative ($\chi_e' = -\omega_p{}^2/\omega^2$). The remaining portion of the propagating momentum density is pure field momentum and equal to

$$(1/v_p)(1 + \chi_e'/2)\epsilon_0 E^2 = \frac{1 + \epsilon_r'}{2v_p} \epsilon_0 E^2 . \tag{31}$$

The total mechanical momentum density is $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0{}^2(-\cos^2\varphi + \sin^2\varphi)$, the $-\cos^2\varphi$ term being associated with the propagating wave and the $\sin^2\varphi$ term with the non-propagating plasma oscillations. The average mechanical momentum density is zero. The total pure field momentum is the sum of the

propagating and non-propagating components, or

$$\frac{1+\epsilon_r{}'}{2v_p}\,\epsilon_0 E_0{}^2\cos^2\varphi + \frac{1-\epsilon_r{}'}{2v_p}\,\epsilon_0 E_0{}^2\cos^2\varphi = \frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi. \qquad (32)$$

The total momentum density is the sum of the pure field momentum and the mechanical momentum, or

$$\begin{aligned}&\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi + \frac{1-\epsilon_r{}'}{2v_p}\,\epsilon_0 E_0{}^2(-\cos^2\varphi+\sin^2\varphi)\\ &= \frac{1+\epsilon_r{}'}{2v_p}\,\epsilon_0 E_0{}^2\cos^2\varphi + \frac{1-\epsilon_r{}'}{2v_p}\,\epsilon_0 E_0{}^2\sin^2\varphi\,; \qquad (32)\end{aligned}$$

its average value is $\epsilon_0 E_0{}^2/2v_p$.

The energy and momentum densities are summarized in Table 2 and illustrated in Figure 5.

The Maxwell stress tensor can be used in either vacuum or macroscopic media form, with different interpretations. The amplitude of the electric field is $E_0 = \sqrt{2I/\epsilon_r{}'\epsilon_0 v_p}$. Using the macroscopic media form, the indicated instantaneous stress on a surface normal to the direction of propagation is

$$\epsilon_r{}'\epsilon_0 E^2/2 + B^2/2\mu_0 = \epsilon_r{}'\epsilon_0 E^2; \qquad (34)$$

this is the radiation pressure, and it has no contribution from the pure field momentum associated with that part of the electric field energy density produced from kinetic energy. Thus, use of the Maxwell stress tensor in macroscopic media form provides no information on the plasma oscillations, although the $B^2/2\mu_0$ term does include the stress associated with the electrical force on the polarization charges; this produces that part of the mechanical momentum that is recognized as emmd through the use of $B^2/2\mu_0 v_p = \epsilon_r{}'\epsilon_0 E^2/2v_p$. Maxwwell's equations recognize the mechanical contribution

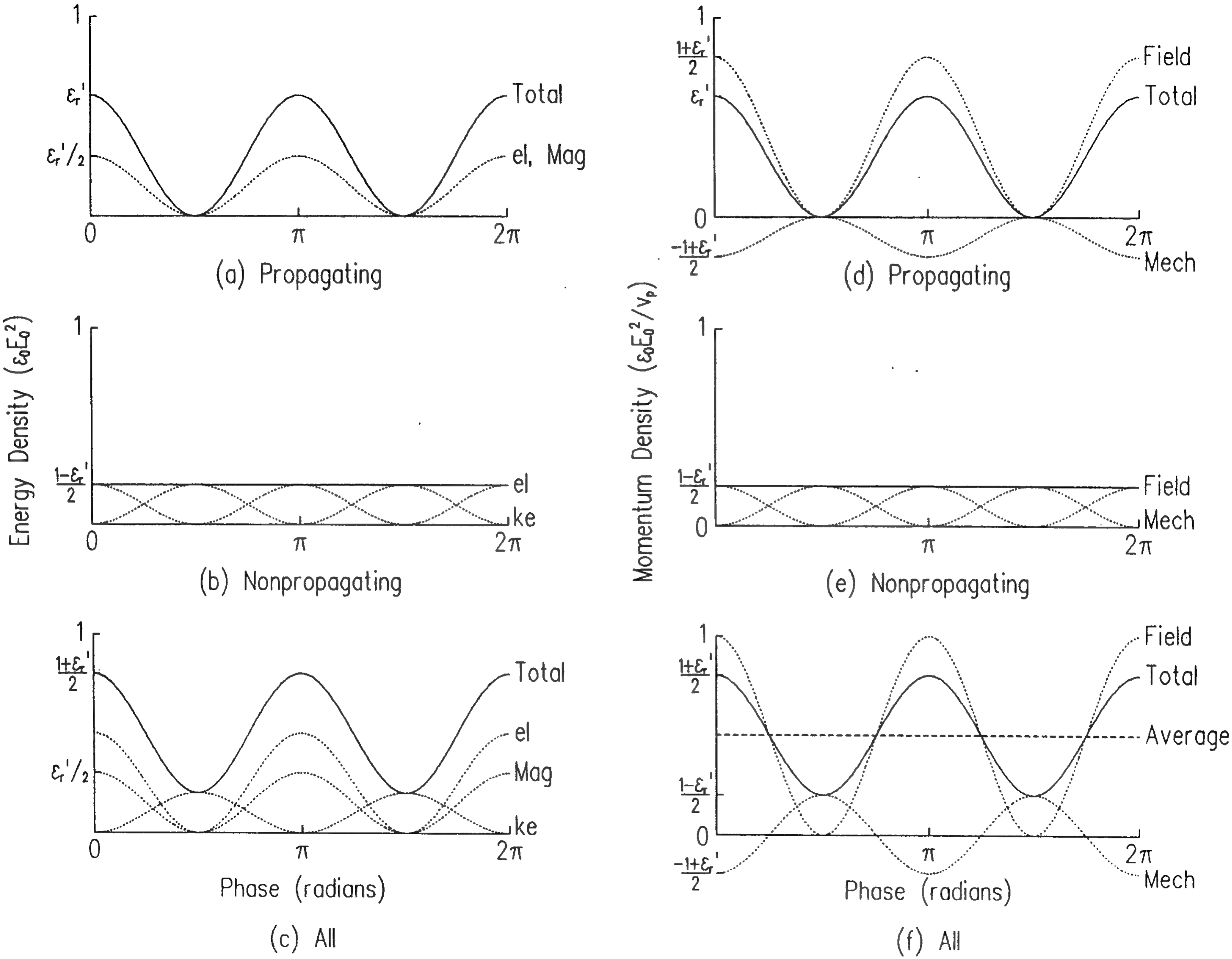

Energy Density ($\varepsilon_0 E_0^2$)
1
ε_r'
$\varepsilon_r'/2$
0
π
2π
Total
el, Mag
(a) Propagating
1
$\frac{1-\varepsilon_r'}{2}$
0
π
2π
el
ke
(b) Nonpropagating
1
$\frac{1+\varepsilon_r'}{2}$
$\varepsilon_r'/2$
0
π
2π
Total
el
Mag
ke
Phase (radians)
(c) All
Momentum Density ($\varepsilon_0 E_0^2/v_p$)
1
$\frac{1+\varepsilon_r'}{2}$
ε_r'
0
$-\frac{1+\varepsilon_r'}{2}$
π
2π
Field
Total
Mech
(d) Propagating
1
$\frac{1-\varepsilon_r'}{2}$
0
π
2π
Field
Mech
(e) Nonpropagating
1
$\frac{1+\varepsilon_r'}{2}$
$\frac{1-\varepsilon_r'}{2}$
0
$-\frac{1+\varepsilon_r'}{2}$
π
2π
Field
Total
Average
Mech
Phase (radians)
(f) All

FIGURE 5

$\chi_e' \epsilon_0 E^2/2$ to the propagating energy density $\epsilon_r' \epsilon_0 E^2/2$ as being electric field energy, and the polarization current acts as effectively as the $\frac{1}{c^2}\frac{\partial E}{\partial t}$ term in determining the value of B).

In vacuum form, the Maxwell stress tensor indicates a stress across a surface normal to the direction of propagation $\epsilon_0 E^2/2 + B^2/2\mu_0 = \frac{1 + \epsilon_r'}{2} \epsilon_0 E^2$, and the stress per unit volume is

$$\frac{\partial}{\partial z}(\epsilon_0 E^2/2 + B^2/2\mu_0) = \frac{1 + \epsilon_r'}{2} \epsilon_0 E_0^2 \, k \sin 2\varphi. \tag{35}$$

A fraction $\frac{1 - \epsilon_r'}{1 + \epsilon_r'}$ of this stress, corresponding to the $-e\,\mathbf{v} \times \mathbf{B}$ forces, acts on the electrons and produces mechanical momentum $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2 \sin^2\varphi$ in the propagation direction, while the remaining fraction $\frac{2\epsilon_r'}{1 + \epsilon_r'}$ corresponds to the flow of momentum with the waves. The $B^2/2\mu_0$ term is equal to $\epsilon_r' \epsilon_0 E^2/2$, and it implicitly involves the electrical force on the electrons that produces mechanical momentum $-\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2 \cos^2\varphi$. Thus the average mechanical momentum is zero.

The association of energy density or stress with the plasma oscillations and the propagating waves is indicated in Figure 6. That part of the electric field energy density associated with the plasma oscillations interchanges cyclically with the kinetic energy density $\frac{1 - \epsilon_r'}{2} \epsilon_0 E_0^2 \sin^2\varphi$ and produces mechanical momentum $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2 \sin^2\varphi$ in so doing. The remainder of the mechanical momentum, $-\frac{1 - \epsilon_r}{2v_p} \epsilon_0 E^2$, propagates with the waves.

TOTAL ELECTROMAGNETIC STRESS IN PLASMAS

E–field

$\epsilon_0 E^2/2$

Plasma osc.

$(1 - \epsilon_r{}')\epsilon_0 E^2/2$

Produces mechanical momentum

$$\frac{1 - \epsilon_r{}'}{2v_p}\,\epsilon_0 E_0{}^2 \sin^2\varphi$$

$\epsilon_r{}'\,\epsilon_0 E^2/2$

B–field

$B^2/2\,\mu_0 =$

Waves

$\epsilon_r{}'\,\epsilon_0 E^2/2$

Radiation Pressure

$\epsilon_r{}'\,\epsilon_0 E^2$

FIGURE 6

The concept of emmd in plasmas is somewhat more straightforward to apply than is the Maxwell stress tensor. Whatever difficulty exists is related to the difficulty encountered in evaluating the electric field energy density and categorizing it as propagating or non-propagating. The quantity $\epsilon_r'\epsilon_0 EB = \epsilon_r'\epsilon_0 E^2/v_p$ identifies the momentum density that propagates with the waves; this could be considered to be the total emmd; it does not include in its definition the non-propagating emmd that is associated with the plasma oscillations, and it does include the negative mechanical momentum density $-\frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E_0^2 \cos^2\varphi$ that propagates with the waves aas part of the emmd. Thus the pure field momentum that propagates with the waves is $\frac{1 + \epsilon_r'}{2v_p}\epsilon_0 E^2$. The non-propagating pure field momentum amounts to $\frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E_0^2\cos^2\varphi$, so the total pure field momentum is $\frac{1}{v_p}\epsilon_0 E_0^2\cos^2\varphi$. The total mechanical momentum is the sum of $\frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E_0^2\sin^2\varphi$ and $-\frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E_0^2 \cos^2\varphi$, the latter term being included in the quantity $\epsilon_r'\epsilon_0 EB$. The total momentum density is the sum of the pure field and the mechanical momentum densities,

$$\begin{aligned}\frac{1}{v_p}\epsilon_0 E_0^2\cos^2\varphi + \frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E_0^2\sin^2\varphi - \frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E_0^2\cos^2\varphi \\ = \frac{1 + \epsilon_r'}{2v_p}\epsilon_0 E_0^2\cos^2\varphi + \frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E_0^2\sin^2\varphi, \qquad (36)\end{aligned}$$

and its average value is $\frac{\epsilon_0 E_0^2}{2v_p} = \frac{\epsilon_r'\epsilon_0 E_0^2}{2v_g} = \frac{I}{v_p v_g} = \frac{I}{c^2}$. The mechanical momentum density contributes nothing to the average.

Alternatively, the emmd could be considered to be given by $\epsilon_0 EB = \epsilon_0 E^2/v_p$, in which case it is necessary to take the mechanical momentum densities into account explicitly. Of the total emmd, a portion $\frac{1 - \epsilon_r'}{2v_p}\epsilon_0 E^2$ is

associated with the plasma oscillations and exchanges cyclically with the mechanical momentum density $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2 \sin^2\varphi$; it does not affect the propagation of the waves. The remainder of the emmd, $\frac{1 + \epsilon_r'}{2 v_p} \epsilon_0 E^2$, propagates with the waves, but it is cancelled in part by the (negative) mechanical momentum $-\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E^2$ that propagates with the waves. The net transport of momentum by the waves is thus

$$\frac{1 + \epsilon_r'}{2 v_p} \epsilon_0 E^2 - \frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E^2 = \frac{\epsilon_r' \epsilon_0 E^2}{v_p} . \tag{37}$$

However, this suffers conceptually from the ad hoc assumption that some of the mechanical momentum propagates with the waves without being a part of the electromagnetic propagation.

DISCUSSION

The definitions of electric and magnetic field energy densities and electromagnetic momentum density in media are rather arbitrary because of the conversions that take place between mechanical and electromagnetic forms. The potential energy in dielectrics is closely coupled to electric field energy and part of it acts as if it were electric field energy. Part of the mechanical momentum also acts as if it were electromagnetic momentum. These facts are conveniently taken into account in dielectrics by accepting DE/2 as the electric field energy density and $\mathbf{D} \times \mathbf{B}$ as the emmd. [Booker (1982) has made particularly clear distinctions among the various forms of the electromagnetic equations appropriate to charges and currents in vacuum,

dielectrics, and partially conducting dielectrics.] The fact that it may be difficult, or at least inconvenient, to evaluate the mechanical properties provides a strong incentive to avoid the necessity of considering them explicitly, but it is difficult to understand the physics involved without at least knowing just how the mechanical properties relate to the pure field quantities. The relationships are sufficiently different in dielectrics and plasmas that they must be considered separately.

In any static system involving electric and magnetic fields (i.e., no sources or sinks of electromagnetic energy), the total linear electromagnetic momentum is zero. However, static systems may possess angular electromagnetic momentum. Systems in which there is a flow of electromagnetic energy from one part of the system to another possess linear electromagnetic momentum; in vacuum, this momentum simply corresponds to the flow of the mass equivalent of the energy from the source to the sink. In media, there are in addition other terms that are best included in the definition of electromagnetic momentum.

Somewhat analogously to the way in which the movement of a charge in an electric field produces additional field energy, the existence of a displacement current in the presence of a constant magnetic field produces electromagnetic momentum (assuming the appropriate direction of motion or current in each case). If a displacement current occurs in a dielectric of relative permitivity χ_e, the displacement current is enhanced by the factor

$1 + \chi_e$, and there is a corresponding momentum enhancement. The momentum enhancement is due to the interaction between the polarization current and the magnetic field, which imparts an impulse to the dielectric; although this impulse is mechanical in nature, it is an inseparable part of the electomagnetic momentum, analogous to the increased energy density in a dielectric relative to vacuum for a given field strength. In the presence of a constant electric field in a dielectric, a rate of change in the magnetic field produces similar momentum density, but in this case the impulse delivered to the dielectric is due to the action of the induced electric field on the polarization charges. If the electric and magnetic fields vary proportionally to one another in time, the two contribute equally to the emmd, including the impulse delivered to the dielectric.

For electromagnetic waves in simple media, non-propagating energy density is the basic cause of the phenomenon of group velocity. A related property of group velocity is the existence of non-propagating momentum density. The non-propagating momentum density in dielectrics is purely mechanical in nature. In plasmas, the non-propagating momentum density is partly electromagnetic and partly mechanical (the two forms interchanging cyclically with one another). In both dielectrics and plasmas, part of the propagating momentum density is mechanical in nature: $\chi_e \epsilon_0 EB$ in dielectrics and $(\chi_e'/2)\epsilon_0 EB$ in plasmas (and χ_e' is negative).

The macroscopic media (Minkowski) form of the stress tensor is conceptually preferable to the vacuum (Abraham) form in dielectrics in that it indicates the stress without any need to attribute to the medium an unrealistic ability to transport a mechanical force field at the velocity of light sufficient to increase the stress exerted by electromagnetic waves from $\epsilon_0 E^2$ to $\epsilon_r \epsilon_0 E^2$. Clearly, it is electromagnetic wave propagation, not transmission by mechanical means, that accounts for the transport of momentum.

For electromagnetic waves in media, the cyclic interchange between electric field energy and kinetic and/or potential energy in the medium causes the momentum flux density associated with an energy flux density I to be I/v_p, rather than I/c as it is in vacuum. The momentum density in vacuum is I/c^2, but in media it is necessary to define just what density is meant – propagating, non-propagating, or total. The average propagating densities for the energy and momentum are I/v_p and $I/v_p{}^2$, respectively. The non-propagating densities are $(\frac{v_p}{v_g} - 1)\ I/v_p$ and $(\frac{v_p}{v_g} - 1)\ I/v_p{}^2$, and the average total densities are I/v_g and $I/v_p v_g$. The instantaneous values for the propagating energy and momentum densities in dielectrics are $S/v_p = \epsilon_r \epsilon_0 E^2$ and $S/v_p{}^2 = \epsilon_r \epsilon_0 E^2/v_p$; for plasmas, it is only necessary to add primes to the ϵ_r factors. That part of the mechanical momentum that propagates with the waves acts just as if it were electromagnetic momentum, in the same way that mechanical energy density in the amount $\chi_e \epsilon_0 E^2/2$ in a dielectric acts just as if it were part of the electric field energy density. However, the amount of propagating mechanical momentum is different in the two cases, $\chi_e \epsilon_0 E^2/v_p$ for

dielectrics and $\chi_e' \epsilon_0 E^2/2v_p$ for plasmas. This is associated with the fact that, in plasmas, the mechanical momentum produced by the electrical force on the polarization chages (i.e., by $\partial B/\partial t$) propagates as part of the electromagnetic wave, while the mechanical momentum produced by the magnetic force on the polarization current (i.e., by $\partial E/\partial t$) does not. In dielectrics, both terms produce mechanical momentum that propagates as part of the waves.

For electromagnetic waves in plasmas, $\epsilon_r' \epsilon_0 E^2/2 = ED/2$ identifies just that part of the electric field energy density that propagates with the waves; $\epsilon_0 E^2/2$ is the total electric field energy density, and $(1 - \epsilon_r')$ is the fraction of it that does not propagate with the waves. The magnetic field energy density, equal to $\epsilon_r' \epsilon_0 E^2/2$, all propagates with the waves. Hence the total energy density that propagates with the waves is $\epsilon_r' \epsilon_0 E^2$, and the non-propagating electric field energy density is $\frac{1 - \epsilon_r'}{2} \epsilon_0 E^2$. The total energy density is the sum of these two, $\frac{1 + \epsilon_r'}{2} \epsilon_0 E^2$, plus the kinetic energy density, $\frac{1 - \epsilon_r'}{2}$ $\epsilon_0 E_0^2 \sin^2\varphi$, or $\epsilon_0 E^2$. The accounting for momentum is similar. The total emmd that propagates with the waves is $\epsilon_r' \epsilon_0 EB = \epsilon_r' \epsilon_0 E^2/v_p$, and it includes a negative mechanical component $\chi_e' \epsilon_0 E^2/2v_p$. Thus the pure field momentum that propagates with the waves is $\frac{1 + \epsilon_r'}{2 v_p} \epsilon_0 E^2$. The non-propagating momentum density associated with the forced plasma oscillations consists of electomagnetic and mechanical components, respectively $\frac{1 - \epsilon_r'}{2v_p}$ $\epsilon_0 E^2$ and $\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0^2 \sin^2\varphi$. Thus the total pure field momentum density (propagating and non-propagating) is $\epsilon_0 E^2/v_p$. The total mechanical

momentum is $-\frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0{}^2 \cos^2\varphi + \frac{1 - \epsilon_r'}{2v_p} \epsilon_0 E_0{}^2 \sin^2\varphi$; its average value is zero; this agrees with Equation (26) since $nn_g = 1$ for plasmas.

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APPENDIX

Here we will evaluate the volume integral P of the emmd $\epsilon_0 \mathbf{E} \times \mathbf{B}$ explicitly for the case of a point charge q located a distance r from the axis of an infinitely long, non–conducting solenoid. The solenoid has radius R, and the magnetic field within it has the constant value $\mathbf{B} = B\mathbf{k}$; outside the solenoid the magnetic field is zero. The electric field inside the solenoid is taken to be the undisturbed electric field of q, ignoring any effects of shielding charges on the surface of the solenoid, as prescribed by Konopinski [1978]. Taking the origin of the coordinate system to coincide with q, we have

$$E_{x,y,z} = \frac{q}{4\pi\epsilon_0} \frac{[x,y,z]}{\sqrt{x^2 + y^2 + z^2}^3} \tag{A1}$$

and the solenoid's axis is along the line $x = 0$, $y = r$. From symmetry, the only non–vanishing component of P integrated over the volume within the solenoid is

$$P_x = \int_\tau \epsilon_0 E_y B_z \, d\tau = \epsilon_0 B \int_{-R}^{R} dx \int_{y_{min}}^{y_{max}} dy \int_{-\infty}^{\infty} dz \, E_y, \tag{A2}$$

where $y_{min} = r - \sqrt{R^2 - x^2}$, $y_{max} = r + \sqrt{R^2 - x^2}$, and E_y is obtained from Equation (A1). Upon the substitution $\tan\theta = z/\sqrt{x^2 + y^2}$, the integration over z reduces to elementary form, with the result that

$$P_x = \frac{qB}{2\pi} \int_{-R}^{R} dx \int_{y_{min}}^{y_{max}} \frac{y}{x^2 + y^2} \, dy. \tag{A3}$$

This in turn is easily integrated to give

$$P_x = \frac{qB}{2\pi}\int_{-R}^{R} \frac{1}{2}\ln\frac{r^2 + R^2 + 2r\sqrt{R^2 - x^2}}{r^2 + R^2 - 2r\sqrt{R^2 - x^2}}\,dx \tag{A4}$$

$$= \frac{qB}{2\pi}2R\int_{0}^{1} \frac{1}{2}\ln\frac{r^2 + R^2 + 2rR\sqrt{1 - x^2}}{r^2 + R^2 - 2rR\sqrt{1 - x^2}}\,dx$$

$$= \frac{qB}{2\pi}2R\int_{0}^{1} \frac{1}{2}\ln\frac{1 + \alpha\sqrt{1 - x^2}}{1 - \alpha\sqrt{1 - x^2}}\,dx, \tag{A5}$$

where
$$\alpha = \alpha(r) = \frac{2\,r\,R}{R^2 + r^2}\,. \tag{A6}$$

Integrating Equation (A5) by parts,

$$P_x = \frac{qBR}{2\pi}\left[x\ln\frac{1 + \alpha\sqrt{1 - x^2}}{1 - \alpha\sqrt{1 - x^2}}\right]_0^1 - \frac{qBR}{2\pi}\int_0^1 x\,d\ln\frac{1 + \alpha\sqrt{1 - x^2}}{1 - \alpha\sqrt{1 - x^2}}$$

$$= 0 + \frac{qBR}{2\pi}\int_0^1 \frac{\alpha x^2}{1 + \alpha\sqrt{1 - x^2}}\left[1 + \frac{1 + \alpha\sqrt{1 - x^2}}{1 - \alpha\sqrt{1 - x^2}}\right]\frac{dx}{\sqrt{1 - x^2}}$$

$$= \frac{qBR}{\pi}\,\alpha\int_0^1 \frac{x^2}{1 - \alpha^2 + \alpha^2 x}\,\frac{dx}{\sqrt{1 - x^2}}. \tag{A7}$$

The integrals appearing in Equations (A5) and (A7) are not found in standard tables. However, the integral in Equation (A7) can be evaluated using the tabulated results of Gradshteyn and Ryzhik [1980]:

$$\int_0^{\pi/2} \ln(1 + a\sin^2 x)\sin^2 x\,dx$$

$$= \frac{\pi}{2}\left(\ln\frac{1 + \sqrt{1 + a}}{2} - \frac{1}{2}\,\frac{1 - \sqrt{1 + a}}{1 + \sqrt{1 + a}}\right) \tag{A8}$$

and

$$\int_0^{\pi/2} \ln(1 + a\sin^2 x)\cos^2 x\,dx$$

$$= \frac{\pi}{2}\left(\ln\frac{1 + \sqrt{1 + a}}{2} + \frac{1}{2}\,\frac{1 - \sqrt{1 + a}}{1 + \sqrt{1 + a}}\right) \tag{A9}$$

for $a > -1$. Adding Equations (A8) and (A9) and differentiating both sides of the result with respect to the parameter a, we find

$$\int_0^{\pi/2} \frac{\sin^2 x}{1 + a \sin^2 x}\, dx = \frac{\pi}{2} \frac{1}{1 + a + \sqrt{1 + a}} . \tag{A10}$$

The substitution $x = \sin^{-1} u$ then yields

$$\int_0^1 \frac{u^2}{1 + au^2} \frac{du}{\sqrt{1 - u^2}} = \frac{\pi}{2} \frac{1}{1 + a + \sqrt{1 + a}} . \tag{A11}$$

After a simple rescaling and redefinition of variables, this becomes

$$\int_0^1 \frac{x^2}{a^2 + b^2 x^2} \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2} \frac{1}{a^2 + b^2 + a\sqrt{a^2 + b^2}} . \tag{A12}$$

The integral appearing in Equation (A7) is a special case of this result, obtained by setting $a^2 = 1 - \alpha^2(r)$ and $b^2 = \alpha^2(r)$. Thus

$$P_x = \frac{qBR}{2} \frac{\alpha(r)}{1 + \sqrt{1 - \alpha^2(r)}} . \tag{A13}$$

$$= \frac{qBR}{2} \frac{2rR}{R^2 + r^2 \pm (R^2 - r^2)}$$

$$= qBr/2, \text{ or } qBR^2/2r. \tag{A14}$$

For $r < R$, symmetry demands that $P_x \to 0$ as $r \to 0$, and the first solution, obtained by choosing the positive square root, is applicable. For $r > R$, it is necessary that $P_x \to 0$ as $r \to \infty$, and the second solution applies.

Table I

ENERGY AND MOMENTUM DENSITIES

FOR ELECTROMAGNETIC WAVES IN DIELECTRICS

ENERGY	MOMENTUM
AVERAGE TOTAL	
$\frac{v_p}{v_g}\frac{\epsilon_r\epsilon_0 E_0{}^2}{2}$	$\frac{\epsilon_r\epsilon_0 E_0{}^2}{2v_g}$
INSTANTANEOUS, PROPAGATING	
$\epsilon_r\epsilon_0 E_0{}^2\cos^2\varphi$ *	$\frac{\epsilon_r\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ $= \epsilon_0 E^2/v_p$ (pure field) $+ \chi_e\epsilon_0 E^2/v_p$ (mechanical)
INSTANTANEOUS, NON-PROPAGATING	
$(\frac{v_p}{v_g} - 1)\frac{\epsilon_r\epsilon_0 E_0{}^2}{2}\sin^2\varphi +$ (k.e.) $(\frac{v_p}{v_g} - 1)\frac{\epsilon_r\epsilon_0 E_0{}^2}{2}\cos^2\varphi$ (p.e.) $= \frac{\omega^2\chi_e}{\omega_0{}^2 - \omega^2}\frac{\epsilon_0 E_0{}^2}{2}$	$(\frac{v_p}{v_g} - 1)\frac{\epsilon_r\epsilon_0 E_0{}^2}{2v_p}$ $= \frac{\omega^2\chi_e}{\omega_0{}^2 - \omega^2}\frac{\epsilon_0 E_0{}^2}{2v_p}$
(kinetic and the corresponding amount of potential energy)	(momentum deposited in the medium **)

* This includes the electric field energy expended to force the oscillation, $\frac{\epsilon_r' - 1}{2}\epsilon_0 E_0{}^2\cos^2\varphi = \chi_e\epsilon_0 E^2/2$, which is also the potential energy in excess of that produced from kinetic energy; it acts like electric field energy and propagates with the waves.

** This momentum is conveyed to the medium during the establishment of the wave field; equal and opposite momentum is conveyed to the medium as the wave field decays, independent of whether or not the momentum has been retained by the medium during the presence of the wave field.

Table II

ENERGY AND MOMENTUM DENSITIES
FOR ELECTROMAGNETIC WAVES IN PLASMAS

ENERGY	MOMENTUM
INSTANTANEOUS TOTAL	
$\frac{\epsilon_0 E_0{}^2}{2}\cos^2\varphi$ (electric) +	$\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ (em) −
$\epsilon_r{}'\frac{\epsilon_0 E_0{}^2}{2}\cos^2\varphi$ (magnetic) +	$\frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ (mech) +
$\frac{1-\epsilon_r{}'}{2}\epsilon_0 E_0{}^2\sin^2\varphi$ (ke)	$\frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\sin^2\varphi$ (mech)
$= \epsilon_r{}'\epsilon_0 E_0{}^2\cos^2\varphi$ (prop, em) +	$= \epsilon_r{}'\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ (prop) +
$\frac{1-\epsilon_r{}'}{2}\epsilon_0 E_0{}^2\cos^2\varphi$ + (non-prop electric)	$\frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ + (non-prop em)
$\frac{1-\epsilon_r{}'}{2}\epsilon_0 E_0{}^2\sin^2\varphi$ (non-prop, kinetic)	$\frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\sin^2\varphi$ (non-prop mech)
PURE FIELD QUANTITIES	
$\epsilon_r{}'\epsilon_0 E_0{}^2\cos^2\varphi$ (prop) +	$\frac{1+\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ (prop) +
$\frac{1-\epsilon_r{}'}{2}\epsilon_0 E_0{}^2\cos^2\varphi$ (non-prop)	$\frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ (non-prop)
$= \frac{1+\epsilon_r{}'}{2}\epsilon_0 E_0{}^2\cos^2\varphi$	$= \frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$
MECHANICAL	
$\frac{1-\epsilon_r{}'}{2}\epsilon_0 E_0{}^2\sin^2\varphi$ (kinetic)	$-\frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\cos^2\varphi$ (prop) +
	$\frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}\sin^2\varphi$ (non-prop)
	$= \frac{1-\epsilon_r{}'}{2}\frac{\epsilon_0 E_0{}^2}{v_p}(-1+2\sin^2\varphi)$
AVERAGE TOTAL	
$\frac{\epsilon_0 E_0{}^2}{2}$	$\frac{\epsilon_0 E_0{}^2}{2v_p}$